Calculating the Beta Function from Three Common Particle Size Distributions

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In many size reduction operations, particle size is described in terms of the log normal or Rosin-Rammler distributions, and to a lesser extent by the normal distribution (Allen, 1981). All of these distributions have the drawback of existing over an infinite range. The log-normal and Rosin-Rammler distributions have the added difficulties of dependent mode and variance, and of allowing skew only to the right. The normal distribution, in contrast, has independent mode and variance, but is useful only for describing symmetric or approximately symmetric particle populations. These difficulties can be eliminated if the distribution is described by a modified version of the beta function (Peleg and Normand, 1986). The beta distribution allows for a mode that can be varied independently of the variance, and also for symmetry and skewness reversal. Since the beta distribution function is best suited to treat normalized size, $0 \le x \le 1$:

$$\chi = \frac{X - X_{\min}}{X_{\max} - X_{\min}} \tag{1}$$

it has a finite range, unlike the normal, log normal and Rosin-Rammler distributions. Since particulates have upper and lower size limits determined by physical considerations, this appears to be another advantage of the beta distribution function.

It has recently been demonstrated through computer simulations (Popplewell et al., 1988) that the normal, log normal and Rosin-Rammler distributions can be approximated by the beta distribution function provided that they are not excessively wide (normalized coefficient of variance on the order of up to about 0.4–0.5), and that the upper and lower size limits can be identified or be reasonably estimated. Such a problem does not exist in experimental particle distributions where the range is determined by the physical constraints on the population.

Since it appears that for real particle populations the beta distribution function can be substituted for the other distributions, it is interesting to demonstrate whether its parameters can be derived directly from their constants. The objectives of this note are to demonstrate that this is indeed possible and to compare such a procedure with nonlinear regression.

Calculation of the Modified Beta Function Constants

The frequency function $f(X)$ or $f(x)$ of the four distributions is given by the following expressions:

the normal distribution, $f_N(X)$

$$f_N(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \left[\frac{X-\mu}{2\sigma^2}\right] \tag{2}$$

the log normal distribution, $f_{\ln}(X)$

$$f_{\ln}(X) = \frac{1}{X_m \ln \sigma g \sqrt{2\pi}} \exp - \left[\frac{\ln^2 \sigma g}{2}\right] \cdot \exp - \left[\frac{(\ln X - \ln X_m)^2}{2 \ln^2 \sigma_g}\right] \tag{3}$$

the Rosin-Rammler distribution $f_{RR}(X)$

$$f_{RR}(X) = nbX^{n-1} \exp - (bX^n) \tag{4}$$

and the modified beta distribution, $f_{am}(x)$

$$f_{am}(x) = \frac{\chi^{am}(1-\chi)^m}{\int_0^1 \chi^{am}(1-\chi)^m \, dx} \tag{5}$$

The expression of these function modes, means, and variances, in terms of their characteristic constants (namely, μ, σ, X_m, σ_g, n, b, a and m) is shown in Table 1.

Once the constants of any distribution are known, the expressions given in the table enable the calculation of the mode, mean, and variance, by substitution. If for any of the first three distributions the real range can be estimated, the mode and mean can be expressed in terms of the normalized size as defined

Table 1. The Characteristic Parameters of the Normal, Log Normal, Rosin-Rammler and the Modified Beta Distributions. (Adapted from Patel et al., 1976).

Distribution	Range*	Mode (X_m or x_m)	Mean (μ)	Variance (σ^2)
Normal Eq. 2	$-\infty < X < \infty$	μ	μ	σ^2
Log Normal** Eq. 3	$0 \le X < \infty$	X_m	$\exp(\theta + \frac{1}{2}s^2)$	$W(W-1)\exp(2\theta)$
Rosin-Rammler Eq. 4	$0 \le X < \infty$	$\left(\frac{n-1}{nb}\right)^{1/n}$	$\frac{\Gamma\left(\frac{n+1}{n}\right)}{b^{1/n}}$	$\frac{\Gamma\left(\frac{n+2}{n}\right) - \left(\Gamma\left(\frac{n+1}{n}\right)\right)^2}{b^{2/n}}$
Modified Beta Eq. 5	$0 \le x \le 1$	$\frac{a}{a+1}$	$\frac{am+1}{(a+1)m+2}$	$\frac{(am+1)(m+1)}{[(a+1)m+2]^2[(a+1)m+3]}$

*Where X is the absolute size and x is the normalized size, Eq. 1.
**Where $s = \ln \sigma_g$, $\theta = \ln X_m + \ln^2\sigma_g$, and $W = \exp(s^2)$.

by Eq. 1, and the normalized variance can be given by

$$\sigma_N^2 = \frac{\sigma^2}{(X_{max} - X_{min})^2} \tag{6}$$

With this definition of the normalized variance, the normalized coefficient of variation is given by

$$COV_N = \frac{\sigma}{\mu - X_{min}} = \frac{\sigma_N}{\mu_N} \tag{7}$$

where μ_N is defined by Eq. 1.

When the modified beta function is substituted for these distributions, it ought to have virtually the same mode, mean, and variance. If so, the constants a and m can be calculated from any two expressions that account for the mode and mean (when they are not the same), variance and mean, or variance and mode. If the mode and mean equations are solved simultaneously, simple expressions for a and m are obtained, while use of the variance and mode, or variance and mean equations requires solution of a cubic polynomial.

The Fit of the Modified Beta Function

Values of the constants a and m of the modified beta function are shown in Table 2, as calculated from the constants of the other distribution functions using the mode-variance expressions. For the normal and Rosin-Rammler distributions, a and m values obtained using the other two methods of calculation (based on the mean mode and mean variance), agreed within about 10% and therefore are not reported. In the case of the log-normal distribution with COV_N less than .5, calculated values of a agreed to within approximately 6%, while the calculated values of m varied up to 30%. These large discrepancies in the value of m should not be thought overly significant, as they occurred at modes where the distribution shape is relatively insensitive to the m value. For comparison, the constants determined by nonlinear regression, in this case using the SPSS software, are also listed.

Table 2 shows that there was good agreement between the calculated and fitted values of a, and to a lesser extent between the magnitudes of m. Figure 1 shows that the curves fitted by nonlinear regression (SPSS) matched the original distribution more closely in the central region than did the curves calculated

Table 2. The Constants of the Modified Beta Function When Applied to Normal, Log Normal and Rosin-Rammler Populations.

Distribution	Characteristic Constants		COV_N	Calc. Constants of the Beta Function: From σ^2 and x_m		By SPSS	
				a	m	a	m
Normal Eq. 2	μ	σ					
	0.5	0.05	0.1	1	48.5	1	49.5
	0.5	0.10	0.2	1	11.0	1	11.6
	0.5	0.15	0.3	1	4.1	1	4.7
	0.5	0.20	0.4	1	1.6	1	2.3
Log Normal Eq. 3	X_m	σ_g					
	20	1.3	0.39	0.35	8.8	0.36	10.9
	20	1.5	0.53	0.30	3.6	0.33	6.4
	20	1.75	0.67	0.19	2.7	0.21	6.7
	20	1.95	0.77	0.14	2.6	0.16	7.9
	40	1.60	0.59	0.24	3.3	0.26	6.4
	60	1.50	0.54	0.31	3.0	0.33	5.5
Rosin-Rammler Eq. 4	n	b					
	2	5×10^{-4}	0.52	0.38	2.9	0.37	3.4
	2.6	9.5×10^{-6}	0.41	0.68	2.8	0.67	3.2
	2.8	8×10^{-6}	0.39	0.77	2.9	0.76	3.2

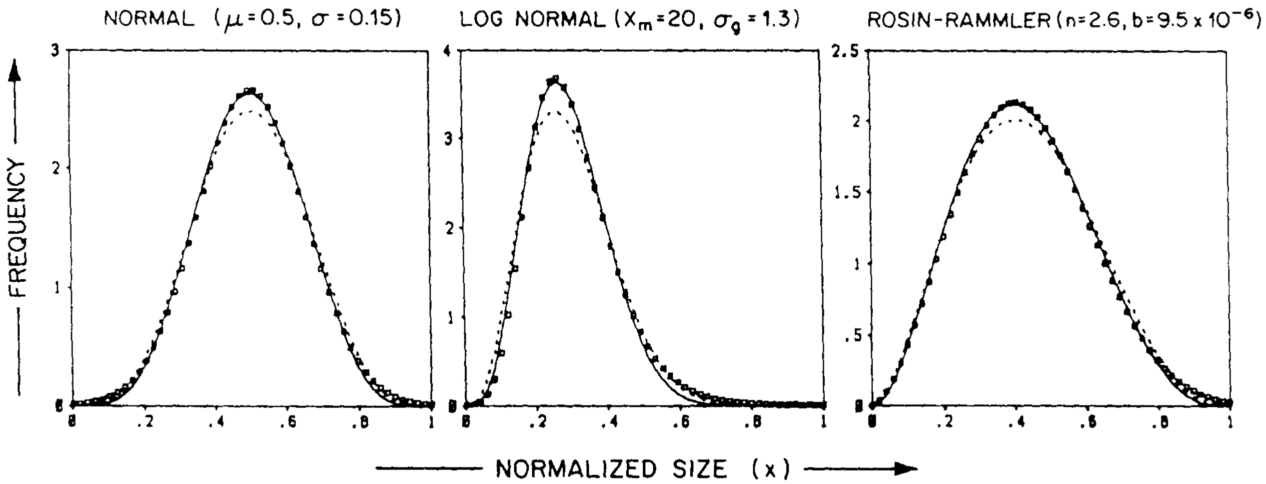

Figure 1. Fit of the beta function to normal, log normal, and Rosin-Rammler populations.
□, generated from data
---, calculated from mode and variance
——, fitted by nonlinear regression (SPSS)

directly from the distribution constants. The situation is reversed at the distribution's edges where the calculated curves were a better match. Since the range of the normal, log normal, and Rosin-Rammler distributions is infinite, their treatment as existing in a finite range creates a discrepancy between their original and normalized mean and variance, but does not affect the mode. This explains why a, the mode related parameter, was much less effected by the calculation procedure than the parameter m.

Thus, with much less effort, direct calculation of the beta function constants yields results comparable to those obtained using non-linear regression.

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Notation

a = power coefficient of the modified beta function, Eq. 5
b = constant of the Rosin-Rammler distribution, Eq. 4
$f_{am}(x)$ = the modified beta function, Eq. 5
$f_N(x)$ = normal distribution, Eq. 2
$f_{ln}(x)$ = log-normal distribution, Eq. 3
$f_{RR}(X)$ = Rosin-Rammler distribution, Eq. 4
m = power coefficient of the modified beta function, Eq. 5
n = power coefficient of the Rosin-Rammler distribution, Eq. 4
x = normalized size
x_m = normalized mode
X = absolute size, length units
X_m = mode, length units
X_{max} = largest absolute size, length units
X_{min} = smallest absolute size, length units
μ = mean size, absolute size units
σ = standard deviation, absolute size units
μ_N = normalized mean size
σ_N = normalized standard deviation

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